

Large-time asymptotic behavior of the infinite system of harmonic oscillators

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Abstract

The mixed initial-boundary value problem for infinite one-dimensional chain of harmonic oscillators on the half-line is considered. We study the large time behavior of solutions and derive the dispersive bounds.

Key words and phrases: one-dimensional system of harmonic oscillators on the half-line, Volterra integro-differential equation, Fourier–Laplace transform, Puiseux expansion, dispersive estimates

1 Introduction

We consider the infinite system of harmonic oscillators on the half-line:

$$\ddot{u}(x, t) = (\Delta_L - m^2)u(x, t), \quad x \in \mathbb{N}, \quad t > 0, \quad (1.1)$$

with the boundary condition (as $x = 0$)

$$\ddot{u}(0, t) = F(u(0, t)) - m^2 u(0, t) - \gamma \dot{u}(0, t) + u(1, t) - u(0, t), \quad t > 0, \quad (1.2)$$

and with the initial condition (as $t = 0$)

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad x \geq 0. \quad (1.3)$$

Here $u(x, t) \in \mathbb{R}$, $m \geq 0$, $\gamma \geq 0$, Δ_L denotes the second derivative on \mathbb{Z} :

$$\Delta_L u(x) = u(x+1) - 2u(x) + u(x-1), \quad x \in \mathbb{Z}.$$

If $\gamma = 0$, then formally the system (1.1)–(1.2) is Hamiltonian with the Hamiltonian functional

$$H(u, \dot{u}) := \frac{1}{2} \sum_{x \geq 0} \left(|\dot{u}(x, t)|^2 + |u(x+1, t) - u(x, t)|^2 + m^2 |u(x, t)|^2 \right) + P(u(0, t)), \quad (1.4)$$

where, by definition, $P(q) := -\int F(q) dq$, $q \in \mathbb{R}$. To prove the existence of solutions to the problem (1.1)–(1.3) we assume that $P \equiv 0$ or

$$P \in C^2(\mathbb{R}), \quad P(q) \rightarrow +\infty \quad \text{as } |q| \rightarrow \infty, \quad (1.5)$$

so $P(q) \geq P_0$ for all q with some $P_0 \in \mathbb{R}$.

Write $Y(t) = (u(\cdot, t), \dot{u}(\cdot, t))$, $Y_0 = (u_0(\cdot), v_0(\cdot))$. We assume that the initial state $Y_0(x)$ belongs to the Hilbert space $\mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$, consisting of real sequences, see Definition 2.1 below. The existence and uniqueness of the solutions $Y(t)$ is proved in Appendix A.

To study the long-time behavior of solutions we assume that $F(q) = -\kappa q$ with $\kappa \geq 0$. Moreover, we impose some restrictions on the coefficients m, κ, γ of the system. Namely, in the case when $\gamma > 0$ we assume that $m > 0$ or $\kappa > 0$. If $\gamma = 0$, then $\kappa \in (0, 2)$. We prove that for any initial state $Y_0 \in \mathcal{H}_{\alpha,+}$ with $\alpha > 3/2$ the solution $Y(t)$ of the system obeys the following bound

$$\|Y(t)\|_{\mathcal{H}_{-\alpha,+}} \leq C(1 + |t|)^{-3/2} \|Y_0\|_{\mathcal{H}_{\alpha,+}}, \quad t \in \mathbb{R}. \quad (1.6)$$

We specify the long-time behavior of the solutions $Y(t)$ in Theorem 2.5.

Harmonic systems with Hamiltonian of a type H_B on the many-dimensional lattice \mathbb{Z}^d , $d \geq 1$, were investigated by many authors, see, for instance, [14, 15]. In [1, 3], these systems were studied with random initial data Y_0 . In [6], we considered the linear coupled "field-particle" system in \mathbb{R}^3 with the field described by continuous Klein–Gordon or wave equations and obtained some results on the long-time behavior for the solutions. The some nonlinear continuous coupled systems were studied by Imaikin, Komech and Vainberg [9] and Jakšić and Pillet [11]. For the solutions of the linear discrete Schrodinger and Klein–Gordon equations, the dispersive estimates of the type (1.6) were obtained by Shaban and Vainberg [18], Komech, Kopylova and Kunze [13] and Pelinovsky and Stefanov [17]. The wave operators for the discrete Schrodinger operators were studied by Cuccagna [2].

2 Main Results

We assume that the initial data Y_0 belongs to the phase space $\mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$, defined below.

Definition 2.1 (i) $\ell_{\alpha,+}^2 \equiv \ell_{\alpha,+}^2(\mathbb{Z}_+)$, $\alpha \in \mathbb{R}$, is the Hilbert space of sequences $u(x)$, $x \geq 0$, with norm $\|u\|_{\alpha,+}^2 = \sum_{x \geq 0} |u(x)|^2 \langle x \rangle^{2\alpha} < \infty$, $\langle x \rangle := (1 + x^2)^{1/2}$.
(ii) $\mathcal{H}_{\alpha,+} = \ell_{\alpha,+}^2 \otimes \ell_{\alpha,+}^2$ is the Hilbert space of pairs $Y = (u, v)$ of sequences equipped with norm $\|Y\|_{\alpha,+}^2 = \|u\|_{\alpha,+}^2 + \|v\|_{\alpha,+}^2 < \infty$.
(iii) $\ell_{\alpha}^2 \equiv \ell_{\alpha}^2(\mathbb{Z})$ is the Hilbert space of sequences with norm $\|u\|_{\alpha}^2 = \sum_{x \in \mathbb{Z}} |u(x)|^2 \langle x \rangle^{2\alpha} < \infty$. In particular, $\ell_0^2 \equiv \ell^2$. Write $\mathcal{H}_{\alpha} := \ell_{\alpha}^2 \otimes \ell_{\alpha}^2$, $\alpha \in \mathbb{R}$.

Theorem 2.2 Let $\gamma, m \geq 0$, condition (1.5) hold, and let $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$. Then the problem (1.1)–(1.3) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$. The operator $U(t) : Y_0 \rightarrow Y(t)$ is continuous on $\mathcal{H}_{\alpha,+}$. Moreover, there exist constants $C, B < \infty$ such that $\|U(t)Y_0\|_{\alpha,+} \leq Ce^{B|t|}$, where the constant C depends on $\|Y_0\|_{\alpha,+}$.

Theorem 2.2 is proved in Appendix A. The proof is based on the following representation for the solution $u(x, t)$ of the problem (1.1)–(1.3):

$$u(x, t) = z(x, t) + q(x, t), \quad x \geq 0, \quad t > 0, \quad (2.1)$$

where $z(x, t)$ is a solution of the mixed problem with zero boundary condition,

$$\ddot{z}(x, t) = (\Delta_L - m^2)z(x, t), \quad x \in \mathbb{N}, \quad t > 0, \quad (2.2)$$

$$z(0, t) = 0, \quad t \geq 0, \quad (2.3)$$

$$z(x, 0) = u_0(x), \quad \dot{z}(x, 0) = v_0(x), \quad x \in \mathbb{N}. \quad (2.4)$$

Therefore, $q(x, t)$ is a solution of the mixed problem with zero initial condition,

$$\ddot{q}(x, t) = (\Delta_L - m^2)q(x, t), \quad x \in \mathbb{N}, \quad t > 0, \quad (2.5)$$

$$\ddot{q}(0, t) = F(q(0, t)) - m^2 q(0, t) - \gamma \dot{q}(0, t) + q(1, t) - q(0, t) + z(1, t), \quad (2.6)$$

$$q(x, 0) = 0, \quad \dot{q}(x, 0) = 0, \quad x \in \mathbb{N}, \quad (2.7)$$

$$q(0, 0) = u_0(0), \quad \dot{q}(0, 0) = v_0(0). \quad (2.8)$$

To prove the main result we assume that $F(q) = -\kappa q$, with $\kappa \geq 0$. Moreover, on the nonnegative constants m, γ, κ of the system we impose the following condition **C**.

C. (i) If $\gamma \neq 0$, then either $m \neq 0$ or $\kappa \neq 0$; (ii) if $\gamma = 0$, then $\kappa \in (0, 2)$.

At first we state the results concerning the solutions of the problem (2.2)–(2.4). Write $Z(t) \equiv Z(x, t) = (z(x, t), \dot{z}(x, t))$.

Lemma 2.3 (see Lemma 2.7 in [5]) Assume that $\alpha \in \mathbb{R}$. Then (i) for any $Y_0 \in \mathcal{H}_{\alpha,+}$, there exists a unique solution $Z(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$ to the mixed problem (2.2)–(2.4); (ii) the operator $U_0(t) : Y_0 \mapsto Z(t)$ is continuous on $\mathcal{H}_{\alpha,+}$. Furthermore, the following bound holds,

$$\|U_0(t)Y_0\|_{\alpha,+} \leq C \langle t \rangle^{\sigma} \|Y_0\|_{\alpha,+}, \quad (2.9)$$

with some constants $C = C(\alpha), \sigma = \sigma(\alpha) < \infty$.

The proof of Lemma 2.3 is based on the following formula for the solution $Z(x, t)$ of the problem (2.2)–(2.4):

$$Z^i(x, t) = \sum_{j=0,1} \sum_{x' \geq 1} \mathcal{G}_{t,+}^{ij}(x, x') Y_0^j(x'), \quad x \in \mathbb{Z}_+, \quad (2.10)$$

where $Z(x, t) = (Z^0(x, t), Z^1(x, t)) \equiv (z(x, t), \dot{z}(x, t))$, the Green function $\mathcal{G}_{t,+}(x, x')$ is

$$\mathcal{G}_{t,+}(x, x') := \mathcal{G}_t(x - x') - \mathcal{G}_t(x + x'), \quad \mathcal{G}_t(x) \equiv \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ix\theta} \hat{\mathcal{G}}_t(\theta) d\theta, \quad (2.11)$$

with

$$\hat{\mathcal{G}}_t(\theta) = (\hat{\mathcal{G}}_t^{ij}(\theta))_{i,j=0}^1 = \begin{pmatrix} \cos \phi(\theta)t & \frac{\sin \phi(\theta)t}{\phi(\theta)} \\ -\phi(\theta) \sin \phi(\theta)t & \cos \phi(\theta)t \end{pmatrix}, \quad \phi(\theta) = \sqrt{2 - 2 \cos \theta + m^2}. \quad (2.12)$$

In particular, $\phi(\theta) = 2|\sin(\theta/2)|$ if $m = 0$. We see that $Z(0, t) \equiv 0 \quad \forall t$, since $\mathcal{G}_t(-x) = \mathcal{G}_t(x)$ (see (2.11) and (2.12)).

For the solutions $U_0(t)Y_0$ of the problem (2.2)–(2.4), the following bound is true.

Theorem 2.4 *Let $Y_0 \in \mathcal{H}_{\alpha,+}$ and $\alpha > 3/2$. Then the following uniform bound holds,*

$$\|U_0(t)Y_0\|_{-\alpha,+} \leq C \langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}, \quad t \in \mathbb{R}. \quad (2.13)$$

This theorem is proved in Appendix C.

To formulate the main result, introduce the following notations.

(i) Write (see (2.11))

$$\begin{aligned} \mathbf{G}_z^i(y, t) &:= \left(\mathcal{G}_{t,+}^{i0}(z, y), \mathcal{G}_{t,+}^{i1}(z, y) \right) \\ &= \left(\mathcal{G}_t^{i0}(z - y) - \mathcal{G}_t^{i0}(z + y), \mathcal{G}_t^{i1}(z - y) - \mathcal{G}_t^{i1}(z + y) \right), \end{aligned} \quad (2.14)$$

$y, z \in \mathbb{Z}$, $i = 0, 1$, $t \in \mathbb{R}$, $\mathcal{G}_t^{ij}(x)$ is defined in (2.11) and (2.12).

(ii) $\mathbf{G}^j(y)$, $j = 0, 1$, denotes the vector valued function defined as

$$\mathbf{G}^j(y) = \int_0^{+\infty} N(s) \mathbf{G}_1^j(y, -s) ds = \int_0^{+\infty} N^{(j)}(s) \mathbf{G}_1^0(y, -s) ds, \quad y \in \mathbb{Z}, \quad (2.15)$$

where $N(s)$ is introduced in (3.12), $\mathbf{G}_1^j(y, s)$ is defined in (2.14).

(iii) Denote by $U'_0(t)$ the operator adjoint to $U_0(t)$:

$$\langle Y, U'_0(t)\Psi \rangle_+ = \langle U_0(t)Y, \Psi \rangle_+, \quad Y \in \mathcal{H}_{\alpha,+}, \quad \Psi \in \mathcal{S} \equiv [S(\mathbb{Z}_+)]^2, \quad t \in \mathbb{R}. \quad (2.16)$$

Here $S(\mathbb{Z}_+)$ denotes the class of rapidly decreasing sequences. In other words,

$$(U'_0(t)\Psi)^j(y) = \sum_{i=0,1} \sum_{x \geq 0} \mathcal{G}_{t,+}^{ij}(x, y) \Psi^i(x), \quad t \in \mathbb{R}, \quad y \in \mathbb{Z}_+, \quad j = 0, 1.$$

In particular, $\mathbf{G}_1^0(y, t) = (U'_0(t)Y_0)(y)$ with $Y_0(x) = (\delta_{1x}, 0)$ (see (2.14)), where δ_{1x} denotes the Kronecker symbol.

(iv) Let $\mathbf{K}^j(x, y)$ $j = 0, 1$, $x \in \mathbb{N}$, $y \in \mathbb{Z}$, stand for vector-valued functions of a form

$$\begin{aligned}\mathbf{K}^j(x, y) &= \int_0^{+\infty} K(x, s) \left(U'_0(-s) \mathbf{G}^j \right) (y) ds \\ &= \int_0^{+\infty} \int_0^{+\infty} K(x, s) N^{(j)}(\tau) \mathbf{G}_1^0(y, -s - \tau) ds d\tau,\end{aligned}\tag{2.17}$$

where $K(x, s)$ is defined in (3.5), \mathbf{G}^j is introduced in (2.15).

(v) Define an operator $\Omega : \mathcal{H}_{0,+} \rightarrow \mathcal{H}_{-\alpha,+}$, $\alpha > 3/2$, by the rule

$$\Omega : Y \rightarrow Y + \left(\langle Y(\cdot), \mathbf{K}^0(x, \cdot) \rangle_+, \langle Y(\cdot), \mathbf{K}^1(x, \cdot) \rangle_+ \right).\tag{2.18}$$

The properties of the functions \mathbf{K}^j and the operator Ω are given in Remark 4.4.

Theorem 2.5 *Let $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha > 3/2$, and condition **C** hold. Then the following assertions are fulfilled.*

(i) $U(t)Y_0 = \Omega(U_0(t)Y_0) + r(t)$, where $\|r(t)\|_{-\alpha,+} \leq C\langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}$.

(ii) The solution $Y(t) = U(t)Y_0$ obeys the following bound

$$\|U(t)Y_0\|_{-\alpha,+} \leq C\langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}, \quad t \in \mathbb{R}.\tag{2.19}$$

This theorem is proved in Section 4.

3 Fourier–Laplace transform

In this section, we study the properties of the solutions $q(x, t)$ to the problem (2.5)–(2.8) using the Fourier–Laplace transform.

Definition 3.1 *Let $|q(t)| \leq Ce^{Bt}$. The Fourier–Laplace transform of $q(t)$ is given by the formula*

$$\tilde{q}(\omega) = \int_0^{+\infty} e^{i\omega t} q(t) dt, \quad \Im \omega > B.\tag{3.1}$$

The Gronwall inequality and conditions on $F(q)$ imply standard a priori estimates for the solutions $q(x, t)$, $x \geq 1$. In particular, there exist constants $A, B < \infty$ such that

$$\sum_{x \in \mathbb{N}} (|q(x, t)|^2 + |\dot{q}(x, t)|^2) \leq Ce^{Bt} \quad \text{as } t \rightarrow +\infty.$$

This bound is proved in Appendix A (see formula (A.18)). Hence the Fourier–Laplace transform with respect to t -variable, $q(x, t) \rightarrow \tilde{q}(x, \omega)$, exists at least for $\Im \omega > B$ and satisfies the following equation

$$(-\Delta_L + m^2 - \omega^2) \tilde{q}(x, \omega) = 0, \quad x \in \mathbb{N}, \quad \Im \omega > B,\tag{3.2}$$

by (2.7). Now we construct the solution of (3.2). We first note that the Fourier transform of the lattice operator $-\Delta_L + m^2$ is the operator of multiplication by the function $\phi^2(\theta) = 2 - 2\cos \theta + m^2$. Thus, $-\Delta_L + m^2$ is a self-adjoint operator and its spectrum is absolutely continuous and coincides with the range of $\phi^2(\theta)$, i.e., $[m^2, m^2 + 4]$. Second, the following lemma holds (see Lemma 2.1 in [13]).

Lemma 3.2 Denote $\Lambda := [-\sqrt{4+m^2}, -m] \cup [m, \sqrt{4+m^2}]$. For given $\omega \in \mathbb{C} \setminus \Lambda$, the equation

$$2 - 2 \cos \theta = \omega^2 - m^2 \quad (3.3)$$

has the unique solution $\theta(\omega)$ in the domain $\{\theta \in \mathbb{C} : \Im \theta > 0, -\pi < \Re \theta \leq \pi\}$. Moreover, $\theta(\omega)$ is an analytic function in $\mathbb{C} \setminus \Lambda$.

Since we seek the solution $q(\cdot, t) \in \ell_{\alpha,+}^2$ with some α , then $\tilde{q}(x, \omega)$ has a form

$$\tilde{q}(x, \omega) = \tilde{q}(0, \omega) e^{i\theta(\omega)x}, \quad x \geq 0.$$

We put $\tilde{K}(x, \omega) = e^{i\theta(\omega)x}$. Applying the inverse Fourier–Laplace transform with respect to ω -variable, we write the solution $q(x, t)$ of (2.5) in the form

$$(q(x, t), \dot{q}(x, t)) = \int_0^t K(x, t-s)(q(0, s), \dot{q}(0, s)) ds, \quad x \in \mathbb{N}, \quad t > 0, \quad (3.4)$$

where

$$K(x, t) = \frac{1}{2\pi} \int_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t} \tilde{K}(x, \omega) d\omega, \quad x \in \mathbb{N}, \quad t > 0, \quad \text{with some } \mu > 0. \quad (3.5)$$

In Appendix A, we study the analytic properties $\tilde{K}(x, \omega)$ for $\omega \in \mathbb{C}$ and obtain the following result.

Theorem 3.3 For any $\alpha < -3/2$, the following bound holds,

$$\sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} |K(x, t)|^2 \leq C(1+t)^{-3} \quad \text{for } t > 0. \quad (3.6)$$

In particular,

$$|K(1, t)| \leq C(1+t)^{-3/2}, \quad t > 0. \quad (3.7)$$

To estimate $q(0, t)$, we again apply the Fourier–Laplace transform. Using (3.4) and equality $F(q) = -\kappa q$ with $\kappa \geq 0$, we rewrite Eqn (2.6) in the form

$$\ddot{q}(0, t) = -(\kappa + 1 + m^2)q(0, t) - \gamma \dot{q}(0, t) + \int_0^t K(1, t-s)q(0, s) ds + z(1, t), \quad t > 0. \quad (3.8)$$

At first, we study the solutions of the corresponding homogeneous equation

$$\ddot{q}(0, t) = -(\kappa + 1 + m^2)q(0, t) - \gamma \dot{q}(0, t) + \int_0^t K(1, t-s)q(0, s) ds, \quad t > 0, \quad (3.9)$$

with the initial data

$$q(0, t)|_{t=0} = u_0(0) =: q_0, \quad \dot{q}(0, t)|_{t=0} = v_0(0) =: p_0. \quad (3.10)$$

Applying the Fourier–Laplace transform to the solutions $q(0, t)$ of (3.9), we obtain

$$\tilde{D}(\omega)\tilde{q}(0, \omega) = -i\omega q_0 + q_0\gamma + p_0 \quad \text{for } \Im \omega > B,$$

where, by definition,

$$\tilde{D}(\omega) := -\omega^2 + \kappa + 1 + m^2 - i\omega\gamma - \tilde{K}(1, \omega), \quad \tilde{K}(1, \omega) = e^{i\theta(\omega)}. \quad (3.11)$$

Write $\tilde{N}(\omega) := [\tilde{D}(\omega)]^{-1}$. Then $\tilde{q}(0, \omega) = \tilde{N}(\omega) (-i\omega q_0 + q_0\gamma + p_0)$. The analytic properties of $\tilde{D}(\omega)$ and $\tilde{N}(\omega)$ for $\omega \in \mathbb{C}$ are studied in Appendix B. In particular, we prove that $\tilde{N}(\omega)$ is analytic in the upper half-space, i.e., with $\Im\omega > 0$. Denote

$$N(t) = \frac{1}{2\pi} \int_{-\infty+i\mu}^{+\infty+i\mu} e^{-i\omega t} \tilde{N}(\omega) d\omega, \quad t \geq 0, \quad \text{with some } \mu > 0. \quad (3.12)$$

The following theorem is proved in Appendix B.

Theorem 3.4 *Let constants m, γ, κ be nonnegative and satisfy condition **C**. Then*

$$|N^{(k)}(t)| \leq C(1+t)^{-3/2}, \quad t \geq 0, \quad k = 0, 1, 2. \quad (3.13)$$

If condition **C** is not satisfied, then $N(t)$ decreases more slowly than $\langle t \rangle^{-3/2}$, see Remark B.9. We need the bound (3.13) to derive asymptotics (4.1) which plays the crucial role in our proof.

Corollary 3.5 *Denote by $S(t)$ a solving operator of the Cauchy problem (3.9), (3.10). Then the variation constants formula gives the following representation for the solution of the problem (3.8), (3.10):*

$$\begin{pmatrix} q(0, t) \\ \dot{q}(0, t) \end{pmatrix} = S(t) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} + \int_0^t S(\tau) \begin{pmatrix} 0 \\ z(1, t - \tau) \end{pmatrix} d\tau, \quad t > 0.$$

Evidently, $S(0) = I$. Moreover, the matrix $S(t)$ has a form $\begin{pmatrix} \dot{N}(t) + \gamma N(t) & N(t) \\ \ddot{N}(t) + \gamma \dot{N}(t) & \dot{N}(t) \end{pmatrix}$. By Theorem 3.4, $|S(t)| \leq C(1+t)^{-3/2}$.

4 Asymptotic behavior of $Y(t)$

At first, we study the asymptotic behavior of the solutions to the problem (3.8), (3.10). Set $q^{(0)}(x, t) = q(x, t)$, $q^{(1)}(x, t) = \dot{q}(x, t)$, $x \in \mathbb{Z}_+$.

Proposition 4.1 *Let $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha > 3/2$, condition **C** hold, and $q(0, t)$ be a solution of the problem (3.8), (3.10). Then*

$$q^{(j)}(0, t) = \langle U_0(t)Y_0, \mathbf{G}^j \rangle_+ + r_j(t), \quad t > 0, \quad |r_j(t)| \leq C\langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}, \quad (4.1)$$

where $j = 0, 1$ and the vector valued functions \mathbf{G}^j are defined in (2.15).

Proof First, Corollary 3.5 and the bound (3.13) imply that

$$\begin{pmatrix} q(0, t) \\ \dot{q}(0, t) \end{pmatrix} = \int_0^t S(\tau) \begin{pmatrix} 0 \\ z(1, t - \tau) \end{pmatrix} d\tau + O((1+t)^{-3/2}), \quad t > 0.$$

Second, $S(\tau) \begin{pmatrix} 0 \\ z(1, t - \tau) \end{pmatrix} = \begin{pmatrix} N(\tau) \\ \dot{N}(\tau) \end{pmatrix} z(1, t - \tau)$. Moreover, for $j = 0, 1$, the bounds (2.13) and (3.13) give

$$\left| \int_t^{+\infty} N^{(j)}(\tau) z(1, t - \tau) d\tau \right| \leq C \int_t^{+\infty} \langle \tau \rangle^{-3/2} \langle t - \tau \rangle^{-3/2} d\tau \leq C \langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}.$$

This implies the representation (4.1), since by (2.14) and (2.10),

$$z(1, t - \tau) = \langle U_0(t) Y_0(\cdot), \mathbf{G}_1^0(\cdot, -\tau) \rangle_+. \quad \blacksquare$$

Remark 4.2 Now we list the properties of the functions $\mathbf{G}_1^i(y, t)$ and \mathbf{G}^i .

(i) By (2.11) and (2.12), $\mathbf{G}_1^i(y, t)$ is odd w.r.t. $y \in \mathbb{Z}$. Formulas (2.12) and the Parseval identity give

$$\|\mathbf{G}_1^0(\cdot, t)\|_0^2 = C \int_{-\pi}^{\pi} \left(\cos^2(\phi(\theta)t) + \frac{\sin^2(\phi(\theta)t)}{\phi^2(\theta)} \right) \sin^2(\theta) d\theta \leq C < \infty. \quad (4.2)$$

(ii) $\mathbf{G}^j(\cdot)$ is odd. Moreover, by the bounds (3.13) and (4.2), $\mathbf{G}^j \in \mathcal{H}_0$, since

$$\|\mathbf{G}^j(\cdot)\|_0 \leq \int_0^{+\infty} |N^{(j)}(s)| \|\mathbf{G}_1^0(\cdot, -s)\|_0 ds \leq C \int_0^{+\infty} |N^{(j)}(s)| ds < \infty. \quad (4.3)$$

Since $U'_0(t) \mathbf{G}_1^0(y, -s) = \mathbf{G}_1^0(y, t - s)$, then (4.3) implies that

$$\sup_{t \in \mathbb{R}} \|U'_0(t) \mathbf{G}^j\|_0 \leq C < \infty. \quad (4.4)$$

Also, it follows from (4.3) that

$$|\langle Y, \mathbf{G}^j \rangle_+| \leq \|Y\|_{0,+} \|\mathbf{G}^j\|_{0,+} \leq C \|Y\|_{0,+}. \quad (4.5)$$

(iii) Since $\mathbf{G}_1^0(y, t) = U'_0(t)(\delta_{1x}, 0)$, then for any $\alpha > 3/2$

$$\|U'_0(t) \mathbf{G}^j\|_{-\alpha,+} \leq \int_0^{+\infty} |N^{(j)}(s)| \|U'_0(t - s)(\delta_{1x}, 0)\|_{-\alpha,+} ds \leq C \langle t \rangle^{-3/2}, \quad (4.6)$$

due to the bounds (3.13) and (2.13), because the action of group $U'_0(t)$ coincides with action of group $U_0(t)$, up to order of the components. Therefore, for $\alpha > 3/2$

$$|\langle U_0(t) Y_0, \mathbf{G}^j \rangle_+| \leq \|Y_0\|_{\alpha,+} \|U'_0(t) \mathbf{G}^j\|_{-\alpha,+} \leq C \langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}. \quad (4.7)$$

Now we study the large time behavior of $q(x, t)$ for $x \in \mathbb{N}$.

Lemma 4.3 Assume that $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha > 3/2$. Then the solution $q(x, t)$, $x \geq 1$, of the problem (2.5)–(2.8) admits the following representation

$$q^{(j)}(x, t) = \langle U_0(t) Y_0, \mathbf{K}^j(x, \cdot) \rangle_+ + r_j(x, t), \quad j = 0, 1, \quad t > 0, \quad (4.8)$$

where \mathbf{K}^j is introduced in (2.17), $\|r_j(\cdot, t)\|_{-\alpha,+} \leq C \langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+}$. Here, by definition, $\|r\|_{\alpha,+}^2 := \sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} |r(x)|^2$.

Proof At first, by (3.4) and (4.1),

$$q^{(j)}(x, t) = \int_0^t K(x, t-s) \langle U_0(s) Y_0, \mathbf{G}^j(\cdot) \rangle_+ ds + r'_j(x, t), \quad (4.9)$$

where $x \in \mathbb{N}$, $\|r'_j(\cdot, t)\|_{-\alpha, +} \leq C \langle t \rangle^{-3/2}$. Indeed, by (4.1) and (3.6),

$$\begin{aligned} \|r'_j(\cdot, t)\|_{-\alpha, +} &= \left\| \int_0^t K(\cdot, t-s) r_j(s) ds \right\|_{-\alpha, +} \leq \int_0^t \|K(\cdot, t-s)\|_{-\alpha, +} |r_j(s)| ds \\ &\leq C \int_0^t (1+t-s)^{-3/2} (1+s)^{-3/2} ds \|Y_0\|_{\alpha, +} \leq C_1 \langle t \rangle^{-3/2} \|Y_0\|_{\alpha, +}. \end{aligned}$$

Second, the first term in the r.h.s. of (4.9) has a form (see (2.17))

$$\int_0^t K(x, s) \langle U_0(t-s) Y_0, \mathbf{G}^j(\cdot) \rangle_+ ds = \langle U_0(t) Y_0, \mathbf{K}^j(x, \cdot) \rangle_+ + r''_j(x, t), \quad (4.10)$$

where, by definition, $r''_j(x, t) = \int_t^{+\infty} K(x, s) \langle U_0(t-s) Y_0, \mathbf{G}^j \rangle_+ ds$. The bounds (3.6) and (4.7) yield

$$\|r''_j(\cdot, t)\|_{-\alpha, +} \leq \int_t^{+\infty} \|K(\cdot, s)\|_{-\alpha, +} |\langle U_0(t-s) Y_0, \mathbf{G}^j \rangle_+| ds \leq C \langle t \rangle^{-3/2} \|Y_0\|_{\alpha, +}. \quad (4.11)$$

Hence, bounds (4.9)–(4.11) imply (4.8) with $r_j(x, t) = r'_j(x, t) + r''_j(x, t)$. ■

Note that if we set $\tilde{K}(0, \omega) := e^{i\theta(\omega)x}|_{x=0} = 1$, then formally, $K(0, t) = \delta_{0t}$. Hence, we can put $\mathbf{K}^j(0, y) = \mathbf{G}^j(y)$, $y \in \mathbb{Z}$. Then, the representation (4.1) follows from (4.8).

Remark 4.4 (i) By Remark 4.2 and (2.17), $\mathbf{K}^j(x, y)$ is odd w.r.t. $y \in \mathbb{Z}$. Furthermore, $\|\mathbf{K}^j(x, \cdot)\|_0 \in \mathcal{H}_{-\alpha, +}$ with $\alpha > 3/2$, since

$$\left\| \|\mathbf{K}^j(x, \cdot)\|_0 \right\|_{-\alpha, +} \leq \int_0^{+\infty} \|K(x, s)\|_{-\alpha, +} \|U'_0(-s) \mathbf{G}^j\|_0 ds < \infty$$

due to (2.17), (3.6) and (4.4). Hence,

$$\|\langle Y_0, \mathbf{K}^j(x, \cdot) \rangle_+\|_{-\alpha, +} \leq C \|Y_0\|_{0, +} \quad \text{for } \alpha > 3/2. \quad (4.12)$$

Therefore, by (2.18) and (4.12),

$$\|\Omega Y\|_{-\alpha, +} \leq \|Y\|_{-\alpha, +} + \sum_{j=0,1} \|\langle Y, \mathbf{K}^j(x, \cdot) \rangle_+\|_{-\alpha, +} \leq C \|Y\|_{0, +}.$$

(ii) By (2.17), (3.6) and (4.6), we have for $\alpha > 3/2$

$$\left\| \|U'_0(t) \mathbf{K}^j(x, \cdot)\|_{-\alpha, +} \right\|_{-\alpha, +} \leq \int_0^{+\infty} \|K(x, s)\|_{-\alpha, +} \|U'_0(t-s) \mathbf{G}^j\|_{-\alpha, +} ds < C \langle t \rangle^{-3/2}.$$

Therefore,

$$\|\langle U_0(t) Y_0, \mathbf{K}^j(x, \cdot) \rangle_+\|_{-\alpha, +} \leq C \langle t \rangle^{-3/2} \|Y_0\|_{\alpha, +} \quad \text{for } \alpha > 3/2. \quad (4.13)$$

Proof of Theorem 2.5 The item (i) follows from the representation (2.1) and the bounds (2.13), (4.1) and (4.8) (if we put in (2.18) $\mathbf{K}^j(0, y) := \mathbf{G}^j(y)$). Further, definition (2.18), the bounds (2.13), (4.7) and (4.13) give

$$\begin{aligned} \|\Omega(U_0(t)Y_0)\|_{-\alpha,+} &\leq \|U_0(t)Y_0\|_{-\alpha,+} + \sum_j \left(|\langle U_0(t)Y_0, \mathbf{G}^j \rangle_+| + \|\langle U_0(t)Y_0, \mathbf{K}^j(x, \cdot) \rangle_+\|_{-\alpha,+} \right) \\ &\leq C\langle t \rangle^{-3/2} \|Y_0\|_{\alpha,+} \end{aligned} \quad (4.14)$$

for $\alpha > 3/2$. Thus, the bound (2.19) follows from the part (i) of Theorem 2.5 and the bound (4.14). \blacksquare

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Appendix A: The behavior of $K(x, t)$ as $t \rightarrow \infty$

In this section, we first study the properties of $\tilde{K}(x, \omega) := e^{i\theta(\omega)x}$ ($\theta(\omega)$ is introduced in Lemma 3.2) applying the technique of [13, 7]. Next, using these properties, we obtain the bound (3.6) for $K(x, t)$. Finally, by the bound (3.6), we prove the existence of solutions of the problem (1.1)–(1.3).

A.1 Properties of $e^{i\theta(\omega)x}$ for $\omega \in \mathbb{C}$ and $x \in \mathbb{N}$

We set $\Lambda := [-\sqrt{4+m^2}, -m] \cup [m, \sqrt{4+m^2}]$, and $\Lambda_0 = \{\pm m, \pm\sqrt{4+m^2}\}$ denotes the set of the “spectral edges”. We indicate the properties **(I)**–**(III)** of the function $\tilde{K}(x, \omega)$ for different values ω such as $\omega \in \mathbb{C} \setminus \Lambda$, $\omega \in \Lambda \setminus \Lambda_0$, and $\omega \in \Lambda_0$.

(I) Let $\omega \in \mathbb{C} \setminus \Lambda$. Then $\Im\theta(\omega) > 0$. In this case, $\tilde{K}(x, \omega)$ exponentially decays in x . Hence, $\tilde{K}(x, \omega)$ is an analytic function in the complex ω -plane with the values in the class $\ell_{\alpha,+}^2$. Moreover, by (3.3) and the condition $\Im\theta(\omega) > 0$, we have

$$|e^{i\theta(\omega)}| \leq C|\omega|^{-2} \quad \text{as } |\omega| \rightarrow \infty. \quad (\text{A.1})$$

Remark A.1 By (3.5) and (A.1), there exist constants $C, B < \infty$ such that

$$\sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} |K(x, t)|^2 \leq C e^{Bt} \quad \text{for any } \alpha \in \mathbb{R}, \quad t > 0. \quad (\text{A.2})$$

(II) Let $\omega \in \Lambda \setminus \Lambda_0$. Then for such ω and for any $x \in \mathbb{N}$, we have the pointwise convergence $\tilde{K}(x, \omega \pm i\varepsilon) \rightarrow \tilde{K}(x, \omega \pm i0)$ as $\varepsilon \rightarrow +0$. Moreover, $|\tilde{K}(x, \omega \pm i\varepsilon)| \leq C$ uniformly in ε . Hence, for any $\alpha < -1/2$ and $\omega \notin \Lambda_0$,

$$\sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} |\tilde{K}(x, \omega \pm i0) - \tilde{K}(x, \omega \pm i\varepsilon)|^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow +0,$$

by the Lebesgue dominated convergence theorem. Furthermore, since $\overline{\theta(\omega)} = -\theta(\bar{\omega})$ for $\omega \in \mathbb{C} \setminus \Lambda$, $\tilde{K}(x, \omega - i0) = \overline{\tilde{K}(x, \omega + i0)}$ for $\omega \in \Lambda \setminus \Lambda_0$ and $x \in \mathbb{N}$.

(III) Now we study the behavior of $\tilde{K}(x, \omega)$ near the points $\omega \in \Lambda_0 = \{\pm m, \pm\sqrt{4+m^2}\}$. Eqn (3.3) implies

$$e^{i\theta(\omega)} = \cos \theta(\omega) + i \sin \theta(\omega) = 1 - \frac{1}{2}(\omega^2 - m^2) + i\sqrt{\omega^2 - m^2} - \frac{1}{4}(\omega^2 - m^2)^2, \quad \omega \in \mathbb{C} \setminus \Lambda. \quad (\text{A.3})$$

The Taylor expansion implies

$$e^{i\theta(\omega)} = 1 + i\sqrt{\omega^2 - m^2} - \frac{1}{2}(\omega^2 - m^2) - \frac{i}{8}(\omega^2 - m^2)^{3/2} + \dots \quad \text{as } \omega \rightarrow \pm m + i0, \quad (\text{A.4})$$

where $\text{sgn}(\Re\sqrt{\omega^2 - m^2}) = \text{sgn}(\Re\omega)$ for $\Im\omega > 0$. This choice of the branch of the complex root $\sqrt{\omega^2 - m^2}$ follows from the condition $\Im\theta(\omega) > 0$. Hence, for $x \in \mathbb{N}$,

$$e^{i\theta(\omega)x} = 1 + \sum_{j=1}^{+\infty} (\omega^2 - m^2)^{j/2} R^j(x), \quad \omega \rightarrow \pm m + i0. \quad (\text{A.5})$$

Here polynomials $R^j(x)$ are of a form $R^j(x) = \sum_{k=1}^j c_k^j x^k$ with coefficients $c_k^j \in \mathbb{C}$, $j \in \mathbb{N}$. For example, $R^1(x) = ix$, $R^2(x) = -x^2/2$, $R^3(x) = -i(4x^3 - x)/24$.

Similarly,

$$e^{i\theta(\omega)} = -1 + i\sqrt{m^2 + 4 - \omega^2} + \frac{1}{2}(m^2 + 4 - \omega^2) - \frac{i}{8}(m^2 + 4 - \omega^2)^{3/2} + \dots \quad (\text{A.6})$$

as $\omega \rightarrow \pm\sqrt{m^2 + 4} + i0$. Here $\sqrt{m^2 + 4 - \omega^2}$ is the complex root and we choose the branch of this root such that $\text{sgn}(\Re\sqrt{m^2 + 4 - \omega^2}) = \text{sgn}(\Re\omega)$ by the condition $\Im\theta(\omega) > 0$. Hence, for $x \in \mathbb{N}$,

$$e^{i\theta(\omega)x} = (-1)^x \left(1 - ix\sqrt{m^2 + 4 - \omega^2} - (m^2 + 4 - \omega^2)x^2/2 + \dots \right) \quad (\text{A.7})$$

as $\omega \rightarrow \pm\sqrt{m^2 + 4} + i0$. If $m = 0$, then (A.3) and the Taylor expansion give

$$e^{i\theta(\omega)} = 1 - \frac{\omega^2}{2} + i\omega\left(1 - \frac{\omega^2}{8} - \frac{\omega^4}{128} + \dots\right) \quad \text{as } \omega \rightarrow 0, \quad (\text{A.8})$$

and $e^{i\theta(\omega)} = -1 + i\sqrt{4 - \omega^2} + \dots$ as $\omega \rightarrow \pm 2 + i0$. Therefore, in the case $m = 0$,

$$e^{i\theta(\omega)x} = \begin{cases} 1 + i\omega x - \omega^2 x^2/2 - i\omega^3(4x^3 - x)/24 + \dots, & \omega \rightarrow 0 \\ (-1)^x(1 - ix\sqrt{4 - \omega^2} - (4 - \omega^2)x^2/2 + \dots), & \omega \rightarrow \pm 2 + i0 \end{cases} \quad (\text{A.9})$$

The representations (A.5), (A.7) and (A.9) imply the following lemma.

Lemma A.2 (cf [13, Lemma 3.2]) For every $N \in \mathbb{N}$,

$$e^{i\theta(\omega)x} = 1 + \sum_{j=1}^N (\omega^2 - m^2)^{j/2} R^j(x) + r_N(\omega, x), \quad \omega \rightarrow \pm m + i0, \quad (\text{A.10})$$

where $\|r_N(\omega, \cdot)\|_{\alpha,+} = O(|\omega^2 - m^2|^{(N+1)/2})$ for $\alpha < -3/2 - N$. Moreover,

$$D_\omega^r(e^{i\theta(\omega)x}) = \sum_{j=1}^N \frac{d^r}{d\omega^r} (\omega^2 - m^2)^{j/2} R^j(x) + \tilde{r}_N(\omega, x), \quad \omega \rightarrow \pm m + i0,$$

where $\|\tilde{r}_N(\omega, \cdot)\|_{\alpha,+} = O(|\omega^2 - m^2|^{(N+1)/2-r})$ for $\alpha < -3/2 - N$. The similar representation holds for $\omega \rightarrow \pm\sqrt{m^2 + 4} + i0$.

Indeed, the bound (A.10) follows from the following representation for remainder (see formula (A.5))

$$r_N(\omega, x) = (\omega^2 - m^2)^{(N+1)/2} \sum_{k=1}^{N+1} b_k(\omega, x) x^k,$$

where $b_k(\omega, x)$ are uniformly bounded for $\omega \rightarrow \pm m + i0$ and any x . In particular,

$$e^{i\theta(\omega)x} = 1 + \sqrt{\omega^2 - m^2} R_0(\omega, x) \quad \text{as } \omega \rightarrow \pm m + i0,$$

where $\sup_{|\omega| \leq m+\delta} |R_0(\omega, x)| \leq C|x|$, and $\sup_{|\omega| \leq m+\delta} \|R_0(\omega, \cdot)\|_{\alpha,+} \leq C < \infty$ for $\alpha < -3/2$.

A.2 Proof of Theorem 3.3

To prove (3.6), we use the properties **(I)**–**(III)** of $\tilde{K}(x, \omega)$. At first, we rewrite $K(x, t)$, $x \in \mathbb{N}$, $t > 0$, in the form

$$K(x, t) = \frac{1}{2\pi} \int_{\Im \omega = \mu > 0} e^{-i\omega t} \tilde{K}(x, \omega) d\omega = -\frac{1}{2\pi} \int_{\Gamma} e^{-i\omega t} \tilde{K}(x, \omega) d\omega, \quad x \in \mathbb{N}, \quad t > 0,$$

where $\Gamma = \{|\omega| = R : R > \sqrt{4 + m^2}\}$, and the contour Γ is oriented anticlockwise. Since $\tilde{K}(x, \omega)$ is analytic in $\mathbb{C} \setminus \Lambda$, we can vary the integration contour on Λ_ε , where Λ_ε surrounds the segments of Λ and belongs to an ε -neighborhood of Λ (Λ_ε is oriented anticlockwise). Taking a limit as $\varepsilon \rightarrow 0$, we find

$$\begin{aligned} K(x, t) &= \frac{1}{2\pi} \int_{\Lambda} e^{-i\omega t} \left(\tilde{K}(x, \omega + i0) - \tilde{K}(x, \omega - i0) \right) d\omega \\ &= \frac{i}{\pi} \int_{\Lambda} e^{-i\omega t} \Im \tilde{K}(x, \omega + i0) d\omega = \sum_{\pm} \sum_{j=1}^2 \frac{i}{\pi} \int_{\Lambda} e^{-i\omega t} P_j^{\pm}(x, \omega) d\omega. \end{aligned} \quad (\text{A.11})$$

Here $P_j^{\pm}(x, \omega) := \zeta_j^{\pm}(\omega) \Im \tilde{K}(x, \omega + i0)$, $j = 1, 2$, where $\zeta_j^{\pm}(\omega)$ are smooth functions such that $\sum_{\pm, j} \zeta_j^{\pm}(\omega) = 1$, $\omega \in \mathbb{R}$, $\text{supp } \zeta_1^{\pm} \subset \mathcal{O}(\pm m)$, $\text{supp } \zeta_2^{\pm} \subset \mathcal{O}(\pm \sqrt{4 + m^2})$ ($\mathcal{O}(a)$ denotes a neighborhood of the point $\omega = a$). In the case $m = 0$, instead of ζ_1^{\pm} (P_1^{\pm}) we introduce the function ζ_1 (respectively, P_1) with $\text{supp } \zeta_1 \subset \mathcal{O}(0)$. By the property **(III)**,

$$\begin{aligned} \|P_1^{\pm}(\cdot, \omega)\|_{\alpha,+} &= O(|\omega \mp m|^{1/2}) \quad \text{if } m \neq 0, \quad \|P_1(\cdot, \omega)\|_{\alpha,+} = O(|\omega|) \quad \text{if } m = 0, \\ \|P_2^{\pm}(\cdot, \omega)\|_{\alpha,+} &= O(|\omega \mp \sqrt{4 + m^2}|) \end{aligned}$$

for any $\alpha < -3/2$. Therefore, using [12, Lemma 10.2] we obtain

$$\left\| \int_{\Lambda} e^{-i\omega t} P_j^{\pm}(x, \omega) d\omega \right\|_{\alpha,+} = O(|t|^{-3/2}), \quad t \rightarrow \infty, \quad \text{for any } \alpha < -3/2. \quad (\text{A.12})$$

The bound (3.6) follows from (A.11) and (A.12). ■

A.3 Existence of solutions

Lemma A.3 *Let $\alpha \in \mathbb{R}$, $m, \gamma \geq 0$, and let P satisfy the condition (1.5). Then the following assertions hold. (i) For every $Y_0 \in \mathcal{H}_{\alpha,+}$, the problem (1.1)–(1.3) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$. Moreover, the operator $U(t) : Y_0 \mapsto Y(t)$, $t \in \mathbb{R}$, is continuous on $\mathcal{H}_{\alpha,+}$, and there exist constants $C, B < \infty$ such that*

$$\|Y(t)\|_{\alpha,+} \leq Ce^{B|t|} \quad \text{for } t \in \mathbb{R}. \quad (\text{A.13})$$

(ii) For $Y_0 \in \mathcal{H}_{0,+}$,

$$H(Y(t)) + \gamma \int_0^t |\dot{u}(0, s)|^2 ds = H(Y_0), \quad t \in \mathbb{R}, \quad (\text{A.14})$$

where $H(Y(t))$ is defined in (1.4). In particular, if $\gamma = 0$, the energy $H(Y(t))$ is conserved and finite.

Proof of Lemma A.3 To prove the existence of $u(x, t)$, it suffices to prove the existence of the solutions $q(t) \equiv q(0, t)$ to the problem (2.6), (2.8). It follows from the representation (2.1), Lemma 2.3 and formula (3.4). Further, using (3.4), we write (2.6) in the equivalent integral form

$$q(t) = \int_0^t \left(\int_0^s \mathcal{F}(\tau, q(\tau)) d\tau \right) ds + \int_0^t \left(\int_0^s z(1, \tau) d\tau - \gamma q(s) \right) ds + C_0 + C_1 t, \quad t > 0. \quad (\text{A.15})$$

Here $\mathcal{F}(t, q(t)) := -(1 + m^2)q(t) + F(q(t)) + \int_0^t K(1, t - s)q(s) ds$, $C_0 = q(0) \equiv q_0$, $C_1 = \dot{q}(0) \equiv p_0$. By the bound (3.7), condition (1.5), and the contraction mapping principle, the solution $q(t)$ of (A.15) is unique on a certain interval $t \in [0, \varepsilon)$ with some $\varepsilon > 0$ depending on the initial data (q_0, p_0) . Hence, by (3.4), the solution $q(x, t)$ of the problem (2.5)–(2.8) is unique on a certain interval $t \in [0, \varepsilon)$ with some $\varepsilon > 0$ depending on the initial data Y_0 . The existence of $z(x, t)$ is stated in Lemma 2.3. This implies the existence of the local solution $u(x, t) = z(x, t) + q(x, t)$ for sufficiently small t . This local solution can be extended to the global solution using the a priori estimate (A.13). Now we verify (A.13). Indeed, by (2.6) and (3.4),

$$\begin{aligned} & \frac{1}{2} \left(|\dot{q}(t)|^2 + (m^2 + 1)|q(t)|^2 \right) + P(q(t)) + \gamma \int_0^t |\dot{q}(s)|^2 ds \\ &= \frac{1}{2} \left(|p_0|^2 + (m^2 + 1)|q_0|^2 \right) + P(q_0) + \int_0^t \dot{q}(s) \left(z(1, s) - \int_0^s K(1, s - \tau)q(\tau) d\tau \right) ds. \end{aligned} \quad (\text{A.16})$$

Define $M(t) = \sup_{0 \leq s \leq t} (|q(s)|^2 + |\dot{q}(s)|^2)$. Then (A.16) and (3.7) yield

$$M(t) \leq C_1 + \int_0^t \sqrt{M(s)} |z(1, s)| ds + C_2 \int_0^t M(s) ds, \quad t > 0.$$

Applying the Gronwall–Bellman integral type inequality (see, for instance, [16]), we find

$$M(t) \leq e^{C_2 t} \left(\sqrt{C_1} + \frac{1}{2} \int_0^t |z(1, s)| e^{C_2 s/2} ds \right)^2, \quad t > 0.$$

Since $|z(1, t)| \leq C\langle t \rangle^\sigma \|Y_0\|_{\alpha,+}$ (see (2.9)), we obtain the a priori bound

$$|q(t)| + |\dot{q}(t)| \leq Ce^{B|t|} \quad (\text{A.17})$$

with some constants $C, B < \infty$. By (A.2) and (3.4),

$$\left(\sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} (|q(x, t)|^2 + |\dot{q}(x, t)|^2) \right)^{1/2} \leq C_1 e^{B|t|}. \quad (\text{A.18})$$

Thus, the a priori bound (A.13) follows from (2.1), (A.17), (A.18) and (2.9). \blacksquare

Remark A.4 Let $F(q) = -\kappa q$ with $\kappa \geq 0$, and $Y_0 \in \mathcal{H}_{0,+}$. Then the energy $H(Y(t))$ is nonnegative and finite, $H(Y(t)) \leq H(Y_0)$ by (A.14).

Appendix B: Properties of $\tilde{D}(\omega)$ and $\tilde{N}(\omega)$ for $\omega \in \mathbb{C}$

In this section, we study $\tilde{D}(\omega)$ and $\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1}$ using the properties (I)–(III) of $\tilde{K}(x, \omega)$. Denote by \mathbb{C}_+ (\mathbb{C}_-) the upper (respectively, lower) half-plane, $\mathbb{C}_\pm = \{\omega \in \mathbb{C} : \pm \Im \omega > 0\}$.

Lemma B.5 (i) $\tilde{N}(\omega)$ is meromorphic for $\omega \in \mathbb{C} \setminus \Lambda$.

(ii) $|\tilde{N}(\omega)| = O(|\omega|^{-2})$ as $|\omega| \rightarrow \infty$.

(iii) $\tilde{D}(\omega) \neq 0$ for all $\omega \in \mathbb{C}_+$.

Proof The first assertion of Lemma B.5 follows from the formula (3.11) and the analyticity of $\tilde{D}(\omega)$ for $\omega \in \mathbb{C} \setminus \Lambda$ (see property (I) of $\tilde{K}(x, \omega)$ in Appendix A). The assertion (ii) follows from (3.11) and (A.1). To prove the third assertion, we assume that $\tilde{D}(\omega_0) = 0$ for some $\omega_0 \in \mathbb{C}_+$. Hence, the function $u_*(x, t) = e^{i\theta(\omega_0)x} e^{-i\omega_0 t}$, $x \geq 0$, $t \geq 0$, is a solution of the problem (1.1)–(1.2) with the initial data $Y_* = e^{i\theta(\omega_0)x} (1, -i\omega_0)$. Therefore, the Hamiltonian (see (1.4)) is

$$H(u_*(\cdot, t), \dot{u}_*(\cdot, t)) = e^{2t \Im \omega_0} H(Y_*) \quad \text{for any } t > 0, \quad \text{where } H(Y_*) > 0.$$

Since $\Im \omega_0 > 0$ and $Y_* \in \mathcal{H}_{0,+}$, this exponential growth contradicts the energy estimate (A.14). Hence, $\tilde{D}(\omega) \neq 0$ for any $\omega \in \mathbb{C}_+$. \blacksquare

Corollary B.6 If $\gamma = 0$, then $\tilde{D}(\omega) \neq 0$ for any $\omega \in \mathbb{C}_-$.

Indeed, if $\gamma = 0$, then $\overline{\tilde{D}(\omega)} = \tilde{D}(\bar{\omega})$, because $\overline{\theta(\omega)} = -\theta(\bar{\omega})$ for $\omega \in \mathbb{C} \setminus \Lambda$. Therefore, Corollary B.6 follows from item (iii) of Lemma B.5.

Now we study the invertibility of $\tilde{D}(\omega + i0)$ for $\omega \in \mathbb{R}$.

Lemma B.7 Let $\omega \in \mathbb{R}$, and the constants γ, m, κ satisfy condition C. Then $\tilde{D}(\omega + i0) \neq 0$ for any $\omega \in \mathbb{R}$.

Proof Step 1: Let $\omega \in \mathbb{R}$ and $|\omega| > \sqrt{4 + m^2}$. Then $\Re \theta(\omega + i0) = \pm \pi$. Therefore,

$$\tilde{D}(\omega + i0) = -\omega^2 + \kappa + 1 + m^2 - i\omega\gamma + e^{-\Im \theta(\omega + i0)} \quad \text{with } \Im \theta(\omega + i0) > 0.$$

Hence, $\Im \tilde{D}(\omega + i0) \neq 0$ iff $\gamma \neq 0$. On the other hand, $\Re \tilde{D}(\omega + i0) = \kappa - 2$ for $\omega = \pm \sqrt{4 + m^2}$ and $\Re \tilde{D}(\omega_1 + i0) < \Re \tilde{D}(\omega_2 + i0)$ if $|\omega_1| > |\omega_2| \geq \sqrt{4 + m^2}$. In particular, $\Re \tilde{D}(\omega + i0) \rightarrow -\infty$

as $|\omega| \rightarrow \infty$. Hence, for $|\omega| > \sqrt{4+m^2}$, $\Re \tilde{D}(\omega + i0) \neq 0$ iff $\kappa \leq 2$. Therefore, for such values ω , $\tilde{D}(\omega + i0) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \leq 2$. If $\gamma = 0$ and $\kappa > 2$, then there exist two points $\pm\omega_0$ ($\omega_0 > \sqrt{4+m^2}$) such that $\tilde{D}(\pm\omega_0 + i0) = 0$.

Step 2: Let $m \neq 0$ and $\omega \in (-m, 0) \cup (0, m)$. For such values ω , $\Re \theta(\omega + i0) = 0$ and $e^{i\theta(\omega+i0)} \in (e^{i\theta(0)}, e^{i\theta(\pm m+i0)}) = (4(m + \sqrt{4+m^2})^{-2}, 1)$. Hence,

$$\Re \tilde{D}(\omega + i0) = -\omega^2 + \kappa + 1 + m^2 - e^{i\theta(\omega+i0)} > \kappa \quad \text{for } |\omega| < m,$$

and $\Re \tilde{D}(\omega + i0) = \kappa$ for $\omega = \pm m$. Therefore, $\tilde{D}(\omega + i0) \neq 0$ for any $|\omega| < m$, since $\kappa \geq 0$.

Step 3: Let $\omega \in (-\sqrt{4+m^2}, -m) \cup (m, \sqrt{4+m^2})$. Then $\Im \theta(\omega + i0) = 0$ and $\Re \theta(\omega + i0) \in (-\pi, 0) \cup (0, \pi)$. Moreover, $\text{sign}(\sin \theta(\omega + i0)) = \text{sign } \omega$. Hence, for such ω and for $m \neq 0$,

$$\begin{aligned} \Im \tilde{D}(\omega + i0) &= -\omega\gamma - \sin \theta(\omega) \\ &= -\text{sign}(\omega) \left(|\omega|\gamma + \sqrt{\omega^2 - m^2} \sqrt{1 - (\omega^2 - m^2)/4} \right). \end{aligned}$$

If $m = 0$, then $\tilde{D}(\omega + i0) = \kappa - \omega^2/2 - i\omega \left(\gamma + \sqrt{1 - \omega^2/4} \right)$. Thus, for such values ω , $\Im \tilde{D}(\omega + i0) \neq 0$ for any $\kappa, \gamma \geq 0$.

Step 4: If $\omega = \pm\sqrt{4+m^2}$, then $e^{i\theta(\omega+i0)} = -1$, and $\tilde{D}(\omega + i0) = \kappa - 2 \mp i\gamma\sqrt{4+m^2}$. Hence,

$$\tilde{D}(\pm\sqrt{4+m^2} + i0) \neq 0 \quad \text{iff } \gamma \neq 0 \quad \text{or } \gamma = 0 \quad \text{and } \kappa \neq 2.$$

If $\omega = \pm m$, then $e^{i\theta(\omega+i0)} = 1$, and $\tilde{D}(\omega + i0) = \kappa \mp i\gamma m$. Hence, in the case $m \neq 0$, $\tilde{D}(\pm m + i0) \neq 0$ iff $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \neq 0$. In the case $m = 0$, $\tilde{D}(0 + i0) = \kappa \neq 0$ iff $\kappa \neq 0$. Lemma B.7 is proved. \blacksquare

Corollary B.8 *Let $m, \gamma, \kappa \geq 0$. If constants m, γ, κ do not satisfy condition **C**, then there exist points $\omega \in \mathbb{R}$ in which $\tilde{D}(\omega + i0)$ vanishes. Namely, if $m = \kappa = 0$, then $\tilde{D}(0 + i0) = 0$ for any $\gamma \geq 0$. If $\gamma = \kappa = 0$, then $\tilde{D}(\pm m + i0) = 0$ for any $m \geq 0$. If $\gamma = 0$ and $\kappa = 2$, then $\tilde{D}(\pm\sqrt{m^2+4} + i0) = 0$ for any m . If $\gamma = 0$ and $\kappa > 2$, then there exists a point $\omega_0 > \sqrt{4+m^2}$ such that $\tilde{D}(\pm\omega_0 + i0) = 0$.*

Now we study the behavior of $\tilde{D}(\omega)$ and $\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1}$ near the points $\omega = \pm m$ and $\omega = \pm\sqrt{4+m^2}$ with any $m, \gamma, \kappa \geq 0$. In the neighborhood of the points $\omega = \pm\sqrt{4+m^2}$ we use the representation (A.6) and obtain

$$\begin{aligned} \tilde{D}(\omega) &\sim \kappa - 2 \mp i\sqrt{4+m^2}\gamma - i(4+m^2-\omega^2)^{1/2} + \frac{1}{2}(4+m^2-\omega^2) - i(\omega \mp \sqrt{4+m^2})\gamma \\ &\quad + \frac{i}{8}(4+m^2-\omega^2)^{3/2} + \dots, \quad \omega \rightarrow \pm\sqrt{4+m^2} + i0, \end{aligned}$$

where $\text{sgn}(\Re\sqrt{m^2+4-\omega^2}) = \text{sgn}(\Re\omega)$. Therefore, if $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \neq 2$, then

$$(\tilde{D}(\omega))^{-1} \sim C_0 + C_1(4+m^2-\omega^2)^{1/2} + \dots, \quad \omega \rightarrow \pm\sqrt{4+m^2} + i0, \quad (\text{B.1})$$

with $C_0 = (\kappa - 2 \mp i\sqrt{4+m^2}\gamma)^{-1}$ and $C_1 = i(\kappa - 2 \mp i\sqrt{4+m^2}\gamma)^{-2}$. If $\gamma = 0$ and $\kappa = 2$, then we have

$$(\tilde{D}(\omega))^{-1} \sim i(4+m^2-\omega^2)^{-1/2} + \frac{1}{2} - \frac{i}{8}(4+m^2-\omega^2)^{1/2} + \dots, \quad \omega \rightarrow \pm\sqrt{4+m^2} + i0. \quad (\text{B.2})$$

In the neighborhood of the points $\omega = \pm m$ we apply (A.4) (if $m \neq 0$) and find

$$\tilde{D}(\omega) \sim \kappa \mp im\gamma - i(\omega^2 - m^2)^{1/2} - i(\omega \mp m)\gamma - \frac{1}{2}(\omega^2 - m^2) + \dots \quad (\text{B.3})$$

as $\omega \rightarrow \pm m + i0$, where $\text{sgn}(\Re\sqrt{\omega^2 - m^2}) = \text{sgn}(\Re\omega)$. In the case $m = 0$, (A.8) yields

$$\tilde{D}(\omega) \sim \kappa - i\omega(\gamma + 1) - \frac{1}{2}\omega^2 + \frac{i}{8}\omega^3 + \dots, \quad \omega \rightarrow 0. \quad (\text{B.4})$$

To study $(\tilde{D}(\omega))^{-1}$ as $\omega \rightarrow \pm m + i0$, we assume first that either (i) $\gamma \neq 0$ and $m \neq 0$; or (ii) $\gamma \neq 0$, $m = 0$ and $\kappa \neq 0$; or (iii) $\gamma = 0$ and $\kappa \neq 0$. Therefore, by (B.3) and (B.4),

$$(\tilde{D}(\omega))^{-1} \sim \begin{cases} 1/\kappa + i\omega(\gamma + 1)/\kappa^2 + \dots, & \omega \rightarrow 0, & \text{if } m = 0, \\ C_0 + C_1(\omega^2 - m^2)^{1/2} + \dots, & \omega \rightarrow \pm m + i0, & \text{if } m \neq 0, \end{cases} \quad (\text{B.5})$$

with $C_0 = (\kappa \mp im\gamma)^{-1}$ and $C_1 = i(\kappa \mp im\gamma)^{-2}$. Secondly, if $\gamma = \kappa = 0$ and $m \neq 0$, then

$$(\tilde{D}(\omega))^{-1} \sim i(\omega^2 - m^2)^{-1/2} - \frac{1}{2} - \frac{i}{8}(\omega^2 - m^2)^{1/2} \dots, \quad \omega \rightarrow \pm m + i0. \quad (\text{B.6})$$

Finally, in the case $m = \kappa = 0$, we have

$$(\tilde{D}(\omega))^{-1} \sim \frac{i}{\omega(\gamma + 1)} - \frac{1}{2(\gamma + 1)^2} - \frac{i\omega(\gamma - 1)}{8(\gamma + 1)^3} + \dots, \quad \omega \rightarrow 0. \quad (\text{B.7})$$

Proof of Theorem 3.4 Using Lemma B.5, we vary the integration contour in (3.12):

$$N(t) = -\frac{1}{2\pi} \int_{|\omega|=R} e^{-i\omega t} \tilde{N}(\omega) d\omega, \quad t > 0, \quad (\text{B.8})$$

where R is chosen enough large such that $\tilde{N}(\omega)$ has no poles in the region $\mathbb{C}_- \cap \{|\omega| \geq R\}$. Note that if $\gamma = 0$, then $\tilde{N}(\omega)$ has no poles in \mathbb{C}_- by Corollary B.6. Denote by σ_j the poles of $\tilde{N}(\omega)$ in \mathbb{C}_- (if they exist). By Lemmas B.5 and B.7, there exists a $\delta > 0$ such that $\tilde{N}(\omega)$ has no poles in the region $\Im\omega \in [-\delta, 0]$. Hence, we can rewrite $N(t)$ as

$$N(t) = -i \sum_{j=1}^K \text{Res}_{\omega=\sigma_j} [e^{-i\omega t} \tilde{N}(\omega)] - \frac{1}{2\pi} \int_{\Lambda_\varepsilon} e^{-i\omega t} \tilde{N}(\omega) d\omega, \quad t > 0,$$

where $\varepsilon \in (0, \delta)$, the contour Λ_ε surrounds segments of Λ and belongs to an ε -neighborhood of Λ (Λ_ε is oriented anticlockwise). Passing to a limit as $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} N(t) &= \frac{1}{2\pi} \int_{\Lambda} e^{-i\omega t} (\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0)) d\omega + o(t^{-N}) \\ &= \sum_{\pm} \sum_{j=1}^2 \frac{1}{2\pi} \int_{\Lambda} e^{-i\omega t} P_j^{\pm}(\omega) d\omega + o(t^{-N}), \quad t \rightarrow +\infty, \quad \text{with any } N > 0. \end{aligned}$$

Here $P_j^{\pm}(\omega) := \zeta_j^{\pm}(\omega)(\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0))$, where $\zeta_j^{\pm}(\omega)$ are as in (A.11). Then (B.1) and (B.5) imply the bound (3.13) with $k = 0$. Here we use the following estimate (with $j = 1/2$)

$$\left| \int_{\mathbb{R}} \zeta(\omega) e^{-i\omega t} (a^2 - \omega^2)^{j/2} d\omega \right| \leq C(1+t)^{-1-j/2} \quad \text{as } t \rightarrow +\infty, \quad j \text{ is odd}, \quad (\text{B.9})$$

$\zeta(\omega)$ is a smooth function, and $\zeta(\omega) = 1$ for $|\omega - a| \leq \delta$ with some $\delta > 0$ (see, for example, [19, Lemma 2]). The bound (3.13) with $k = 1, 2$ can be proved by a similar way. \blacksquare

Remark B.9 Now we study the asymptotics of $N(t)$ in the case when condition **C** is not fulfilled. Assume first that $\gamma = \kappa = 0$ and $m \neq 0$. Then $\tilde{N}(\omega+i0) - \tilde{N}(\omega-i0) = 2i\Im \tilde{N}(\omega+i0)$, and $\tilde{N}(\omega)$ has no poles in \mathbb{C}_- . Introduce the circles c_\pm , $c_\pm = \{|\omega \mp m| = \varepsilon\}$, with some $\varepsilon \in (0, m)$. We change the integration contour in (B.8) on $\Gamma_\varepsilon := \cup_\pm c_\pm \cup_j \gamma_j$, where γ_j , $j = 1, 2, 3$, stand for the segments of the real axis connecting the circles c_\pm and passing in two directions, $\gamma_1 = [-\sqrt{m^2+4}, -m-\varepsilon]$, $\gamma_2 = [-m+\varepsilon, m-\varepsilon]$, $\gamma_3 = [m+\varepsilon, \sqrt{m^2+4}]$. Using the Cauchy theorem and Lemma B.5, we find

$$N(t) = -\frac{1}{2\pi} \int_{c_- \cup c_+} e^{-i\omega t} \tilde{N}(\omega) d\omega + \sum_{j=1}^3 \frac{i}{\pi} \int_{\gamma_j} e^{-i\omega t} \Im \tilde{N}(\omega+i0) d\omega.$$

Applying representations (B.1) and (B.6) and the well-known estimate (see, for example, [19])

$$-\frac{1}{2\pi} \int_{|\omega|=m+1} e^{-i\omega t} (\omega^2 - m^2)^{-1/2} d\omega = \sqrt{\frac{2}{\pi m t}} i \cos(mt - \pi/4) + O(t^{-3/2}), \quad t \rightarrow \infty,$$

we obtain $N(t) = -(2/(\pi m))^{1/2} t^{-1/2} \cos(mt - \pi/4) + O(t^{-3/2})$ as $t \rightarrow \infty$. Similarly, if $\gamma = 0$ and $\kappa = 2$, then (B.2) and (B.5) give the bound $|N(t)| \leq C\langle t \rangle^{-1/2}$.

In the case $\gamma = 0$ and $\kappa > 2$, $\tilde{N}(\omega)$ has two simple poles in the points $\pm\omega_0$ with $\omega_0 > \sqrt{4+m^2}$. Then, calculating the rescue of $e^{-i\omega t} \tilde{N}(\omega)$ in these points, we have $N(t) \sim C_1 \sin \omega_0 t + O(t^{-3/2})$ as $t \rightarrow +\infty$.

Finally, in the case $m = \kappa = 0$, we use formulas (B.7) and (B.1), calculate the rescue of $\tilde{N}(\omega)$ in the point $\omega = 0$ and obtain $N(t) = (\gamma + 1)^{-1} + O(t^{-3/2})$ as $t \rightarrow +\infty$.

Appendix C: Zero boundary condition

Consider the following mixed initial-boundary value problem on the half-line:

$$\begin{cases} \ddot{z}(x, t) = (\Delta_L - m^2)z(x, t), & x \in \mathbb{N}, \quad t \in \mathbb{R}, \\ z(0, t) = 0, \\ z(x, 0) = u_0(x), \quad \dot{z}(x, 0) = v_0(x), & x \in \mathbb{N}. \end{cases} \quad (\text{C.1})$$

Without loss of generality, we assume that $u_0(0) = v_0(0) = 0$.

Write $Z(x, t) = (Z^0(x, t), Z^1(x, t)) \equiv (z(x, t), \dot{z}(x, t))$, $Y_0(x) = (u_0(x), v_0(x))$. The solution of problem (C.1) can be represented as the restriction of the solution to the Cauchy problem with odd initial data on the half-line,

$$Z^i(x, t) = \sum_{y \in \mathbb{Z}} \mathcal{G}_t^{ij}(x - y) Y_{\text{odd}}^j(y), \quad x \geq 0, \quad i = 0, 1. \quad (\text{C.2})$$

Here $\mathcal{G}_t(x)$ is defined in (2.11) and (2.12), and by definition,

$$Y_{\text{odd}}(x) = Y_0(x) \quad \text{for } x > 0, \quad Y_{\text{odd}}(0) = 0, \quad Y_{\text{odd}}(x) = -Y_0(-x) \quad \text{for } x < 0. \quad (\text{C.3})$$

To prove Theorem 2.4 we first consider the following Cauchy problem for the discrete Klein–Gordon equation in the whole line,

$$\begin{cases} \ddot{u}(t, x) = (\Delta_L - m^2)u(t, x), & t \in \mathbb{R}, \quad x \in \mathbb{Z}, \\ u(t, x)|_{t=0} = u_0(x), \quad \dot{u}(t, x)|_{t=0} = v_0(x). \end{cases} \quad (\text{C.4})$$

It is well-known (see for instance, [14]), that for any $Z_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha$, there exists a unique solution $W_t Z_0 \in C(\mathbb{R}, \mathcal{H}_\alpha)$ to the problem (C.4). Moreover, there exist constants $C, \sigma = \sigma(\alpha) < \infty$ such that the following bound holds,

$$\|W_t Z_0\|_\alpha \leq C \langle t \rangle^\sigma \|Z_0\|_\alpha, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R}. \quad (\text{C.5})$$

Lemma C.1 *Let $Z_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha$ with $\alpha > 5/2$. If $\hat{Z}_0(0) = \hat{Z}_0(\pi) = 0$, then*

$$\|W_t Z_0\|_{-\alpha} \leq C \langle t \rangle^{-3/2} \|Z_0\|_\alpha, \quad t \in \mathbb{R}. \quad (\text{C.6})$$

Otherwise, $\|W_t Z_0\|_{-\alpha} \leq C \langle t \rangle^{-1/2} \|Z_0\|_\alpha$, $t \in \mathbb{R}$.

Below we outline the proof of this lemma.

C.4 Properties of the resolvent of the discrete Klein–Gordon operator

Consider the asymptotic properties of the solutions $u(x, t)$, $x \in \mathbb{Z}$, $t \geq 0$, to the problem (C.4) using the Fourier–Laplace transform. By the bound (C.5), the Laplace–Fourier transform (3.1) with respect to t -variable, exists at least for $\Im \omega > 0$ and satisfies the following equation (3.2) for $x \in \mathbb{Z}$, $\Im \omega > 0$.

Let u be a solution of the equation $(-\Delta_L + m^2 - \omega^2)u = f$ with $f \in \ell^2$. Define the resolvent operator R_ω as $u = R_\omega f = (-\Delta_L + m^2 - \omega^2)^{-1} f$.

Applying the inverse Fourier–Laplace transform with respect to ω -variable, we write the solution $u(x, t)$ of the problem (C.4) in the form

$$u(x, t) = \frac{1}{2\pi} \int_{\Im \omega = \mu} e^{-i\omega t} R_\omega (v_0(x) - i\omega u_0(x)) d\omega, \quad x \in \mathbb{Z}, \quad t > 0, \quad \mu > 0. \quad (\text{C.7})$$

The integral in (C.7) is understood in the sense of principal value. To study the large time behavior of $u(x, t)$, we list properties **I–V** of the resolvent operator R_ω for $\omega \in \mathbb{C}$, see [10, 18, 13]. To formulate them we denote by $B(\alpha, \alpha') = \mathcal{L}(\ell_\alpha^2, \ell_{\alpha'}^2)$ the space of bounded linear operators from ℓ_α^2 to $\ell_{\alpha'}^2$, and by $\text{Op}(R(x, y))$ the operator with the kernel $R(x, y)$, $x, y \in \mathbb{Z}$. As before we write $\Lambda = [-\sqrt{4+m^2}, -m] \cup [m, \sqrt{4+m^2}]$, and $\Lambda_0 = \{\pm m, \pm\sqrt{4+m^2}\}$.

I. For $\omega \in \mathbb{C} \setminus \Lambda$, the resolvent R_ω is the integral operator with the kernel $R_\omega(x, y)$,

$$R_\omega(x, y) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{-i\theta(x-y)}}{2 - 2\cos \theta + m^2 - \omega^2} d\theta, \quad x, y \in \mathbb{Z}.$$

By the Cauchy Residue Theorem, we have

$$R_\omega(x, y) = -i \frac{e^{-i\theta(\omega)|x-y|}}{2 \sin(\theta(\omega))}, \quad x, y \in \mathbb{Z}, \quad \omega \in \mathbb{C} \setminus \Lambda, \quad (\text{C.8})$$

where $\theta(\omega)$ is defined in Lemma 3.2.

II. For $\omega \in \mathbb{C} \setminus \Lambda$, the resolvent R_ω is analytic in the complex ω -plane with the cut along the intervals Λ . Moreover, the sequence $\{e^{-i\theta(\omega)|x|}\}$, $x \in \mathbb{Z}$, is exponentially decaying as

$|x| \rightarrow \infty$. Hence for $\omega \in \mathbb{C} \setminus \Lambda$, R_ω is a bounded operator in $\ell^2(\mathbb{Z})$.

III. Put $\theta(\omega \pm i0) := \lim_{\varepsilon \rightarrow +0} \theta(\omega \pm i\varepsilon)$. For $\omega \in \Lambda \setminus \Lambda_0$ and $x, y \in \mathbb{Z}$, the following pointwise limit exists $R_{\omega \pm i\varepsilon}(x, y) \rightarrow R_{\omega \pm i0}(x, y)$ as $\varepsilon \rightarrow +0$. Moreover, $|\theta(\omega \pm i\varepsilon)| \leq C(\omega)$ and $|\sin \theta(\omega)| > 0$ for $\omega \in \Lambda \setminus \Lambda_0$. Hence, $|R_{\omega \pm i\varepsilon}(x, y)| \leq C(\omega)$ for $\omega \in \Lambda \setminus \Lambda_0$. Therefore, for any $\alpha > 1/2$ and $\omega \notin \Lambda_0$,

$$\sum_{x, y \in \mathbb{Z}} |R_{\omega \pm i\varepsilon}(x, y) - R_{\omega \pm i0}(x, y)|^2 \langle x \rangle^{-2\alpha} \langle y \rangle^{-2\alpha} \rightarrow 0, \quad \varepsilon \rightarrow +0,$$

by the Lebesgue dominated convergence theorem. Thus, for $\omega \in \Lambda \setminus \Lambda_0$, the resolvent $R_{\omega \pm i\varepsilon}$ converges to $R_{\omega \pm i0}$ ($\varepsilon \rightarrow +0$) as Hilbert–Schmidt operators in the space $B(\alpha, -\alpha)$, $\alpha > 1/2$. Moreover, by the Sokhotsky–Plemely formula,

$$\begin{aligned} R_{\omega \pm i0}(x, y) &= \lim_{\varepsilon \rightarrow +0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\theta(x-y)}}{2 - 2\cos \theta + m^2 - (\omega \pm i\varepsilon)^2} d\theta \\ &= \pm \frac{\pi i}{2\pi} \frac{e^{-i\theta(\omega)|x-y|}}{2|\sin(\theta(\omega))|} + \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \frac{e^{-i\theta(x-y)}}{2 - 2\cos \theta + m^2 - \omega^2} d\theta, \end{aligned}$$

where the last integral being taken in the principal value sense. Note also that $\|R_{\omega \pm i0}f\|_{\ell^\infty} \leq C\|f\|_{\ell^1}$, since $|\sin \theta(\omega \pm i0)| > 0$.

IV. $\overline{\theta(\omega)} = -\theta(\bar{\omega})$ for $\omega \in \mathbb{C} \setminus \Lambda$. Hence, $R_{\omega - i0}(x, y) = \overline{R_{\omega + i0}(x, y)}$ for $\omega \in \Lambda \setminus \Lambda_0$, $x, y \in \mathbb{Z}$.

V. The operator $R_{\omega \pm i0}$ diverges near points $\omega \in \Lambda_0$ because $\sin \theta(\omega + i0)$ vanishes in these points. By direct calculation, we obtain a formal Puiseux expansion of R_ω as $\omega \rightarrow \omega_0 + i0$, $\omega_0 \in \Lambda_0$. Namely, for $\omega \rightarrow \pm m + i0$ ($m \neq 0$), we have

$$R_\omega(x, y) = \frac{i}{2}(\omega^2 - m^2)^{-1/2} - \frac{1}{2}|x - y| - \frac{i}{16}(4|x - y|^2 - 1)(\omega^2 - m^2)^{1/2} + \dots,$$

where the branch of the complex root $\sqrt{\omega^2 - m^2}$ is chosen from the condition $\Im \sqrt{\omega^2 - m^2} > 0$. This choice follows from the condition $\Im \theta(\omega) < 0$. Therefore, if $m \neq 0$,

$$R_\omega(x, y) \sim \sum_{j=-1}^{\infty} (\omega^2 - m^2)^{j/2} R_j(x, y), \quad \omega \rightarrow \pm m + i0, \quad (\text{C.9})$$

where $R_{-1}(x, y) = i/2$, $R_0(x, y) = -(1/2)|x - y|$, $R_1(x, y) = -i(|x - y|^2 - 1)/16$, $R_2(x, y) = (|x - y|^3 - |x - y|)/12$, $R_j(x, y) = \sum_{k=0}^{j+1} c_{kj}|x - y|^k$ with some $c_{kj} \in \mathbb{C}$, $j \geq -1$. In the case $m = 0$,

$$R_\omega(x, y) \sim \sum_{j=-1}^{\infty} \omega^j R_j(x, y), \quad \omega \rightarrow 0,$$

with the same $R_j(x, y)$ as in (C.9). For $\omega \rightarrow \pm \sqrt{4 + m^2} + i0$,

$$R_\omega(x, y) = (-1)^{|x-y|} \left(\frac{i}{2}(4 + m^2 - \omega^2)^{-1/2} + \frac{1}{2}|x - y| - \frac{i}{16}(4|x - y|^2 - 1)\sqrt{4 + m^2 - \omega^2} + \dots \right),$$

where $\Im \sqrt{4 + m^2 - \omega^2} > 0$. Applying the following estimate to the terms $R_j(x, y)$,

$$\sum_{x, y \in \mathbb{Z}} \langle x \rangle^{-2\alpha} |x - y|^{2p} \langle y \rangle^{-2\alpha} < \infty \quad \text{for } \alpha > \frac{1}{2} + p, \quad \text{with any } p = 0, 1, 2, \dots,$$

we come to the following result.

Lemma C.2 For $\omega \rightarrow \pm m + i0$, the resolvent R_ω has the expansion

$$R_\omega = \sum_{j=-1}^N (\omega^2 - m^2)^{j/2} R_j + r_N(\omega), \quad \omega \rightarrow \pm m + i0, \quad (\text{C.10})$$

the operators $R_j = \text{Op}(R_j(x, y)) \in B(\alpha, -\alpha)$ for $\alpha > j + 3/2$. This asymptotics holds in the operator sense, the remainder term $r_N(\omega)$ is estimated as $\|r_N(\omega)\|_{B(\alpha, -\alpha)} = O(|\omega^2 - m^2|^{(N+1)/2})$ with $\alpha > 5/2 + N$. The similar decomposition is true for $\omega \rightarrow \pm\sqrt{4+m^2} + i0$.

This lemma can be proved similarly as Lemma 3.2 in [13].

Corollary C.3 (i) Let $f \in \ell_\alpha^2$, $\alpha > 5/2$. For $\omega \rightarrow \pm m + i0$, we have

$$(R_\omega f)(x) = \frac{i\hat{f}(0)}{2\sqrt{\omega^2 - m^2}} - \frac{1}{2} \sum_{y \in \mathbb{Z}} |x - y| f(y) + r_1(\omega) f,$$

where $\|r_1(\omega) f\|_{-\alpha} \leq C|\omega^2 - m^2|^{1/2} \|f\|_\alpha$. For $\omega \rightarrow \pm\sqrt{4+m^2} + i0$,

$$(R_\omega f)(x) = \frac{i(-1)^x \hat{f}(\pi)}{2\sqrt{4+m^2 - \omega^2}} + \frac{1}{2} \sum_{y \in \mathbb{Z}} (-1)^{|x-y|} |x - y| f(y) + r_2(\omega) f,$$

where $\|r_2(\omega) f\|_{-\alpha} \leq C|4+m^2 - \omega^2|^{1/2} \|f\|_\alpha$.

(ii) If $\hat{f}(0) = \hat{f}(\pi) = 0$, then

$$(\mathfrak{S} R_\omega f)(x) \sim C\sqrt{\omega^2 - m^2} \sum_{y \in \mathbb{Z}} (4|x - y|^2 - 1) f(y) \quad \text{as } \omega \rightarrow \pm m + i0,$$

$$(\mathfrak{S} R_\omega f)(x) \sim C\sqrt{4+m^2 - \omega^2} \sum_{y \in \mathbb{Z}} (-1)^{|x-y|} (4|x - y|^2 - 1) f(y) \quad \text{as } \omega \rightarrow \pm\sqrt{4+m^2} + i0.$$

C.5 Proof of the bound (2.13)

To prove (2.13), we apply first Corollary C.3 (ii) and the representation (C.2). Let $Y_0 \in \mathcal{H}_{\alpha,+}$ with $\alpha > 3/2$. Then $Y_{\text{odd}} = (u_{\text{odd}}, v_{\text{odd}}) \in \mathcal{H}_\alpha$ and $\hat{Y}_{\text{odd}}(0) = \hat{Y}_{\text{odd}}(\pi) = 0$ (see (C.3)). Hence, we can apply Corollary C.3 (ii) to $f_{\text{odd}}(x) := v_{\text{odd}}(x) - i\omega u_{\text{odd}}(x)$. Second,

$$\sum_{y \in \mathbb{Z}} |x - y|^2 f_{\text{odd}}(y) = \sum_{y \in \mathbb{Z}_+} (|x - y|^2 - |x + y|^2) f_0(y) = -2x \sum_{y \in \mathbb{Z}_+} y f_0(y),$$

where $f_0(x) := v_0(x) - i\omega u_0(x)$, $x \in \mathbb{Z}_+$. Hence, for $\alpha > 3/2$,

$$\begin{aligned} \left\| \sum_{y \in \mathbb{Z}} (4|x - y|^2 - 1) f_{\text{odd}}(y) \right\|_{-\alpha,+}^2 &\leq C \sum_{x \in \mathbb{Z}_+} \langle x \rangle^{-2\alpha} |x|^2 \left| \sum_{y \in \mathbb{Z}_+} y f_0(y) \right|^2 \\ &\leq C \left| \sum_{y \in \mathbb{Z}_+} \langle y \rangle^{-\alpha} |y| \cdot \langle y \rangle^\alpha |f_0(y)| \right|^2 \leq C \|f_0\|_{\alpha,+}^2. \end{aligned}$$

Thus, we obtain for $\alpha > 3/2$

$$\|\mathfrak{S} R_\omega f_{\text{odd}}\|_{-\alpha,+} \sim \begin{cases} C\sqrt{\omega^2 - m^2} \|f_0\|_{\alpha,+}, & \omega \rightarrow \pm m + i0, \\ C\sqrt{4+m^2 - \omega^2} \|f_0\|_{\alpha,+}, & \omega \rightarrow \pm\sqrt{4+m^2} + i0. \end{cases} \quad (\text{C.11})$$

Further, we use the inverse Fourier-Laplace transform, the bound (C.11) and apply the resonings similar to the proof of Theorem 3.3 (see Section A.2) or to the proof of Theorem 3.4 (see Appendix B), and use the bound (B.9). ■

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